Axial Algebras – recent research and problems

Justin M^{c}Inroy

Heilbronn Institute for Mathematical Research
University of Bristol

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We will now discuss in a bit less detail some other concepts and give some current (and future) research directions.

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The results on sum decompositions are from:

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If the sum $u = \sum_{i \in I} u_i$, where $u_i \in A_i$, is unique for all $\in A$, then we say that the sum is direct and write $A = \boxplus_{i \in I} A_i$.

Lemma

If $A = \square_{i \in I} A_i$ and $\mathrm{Ann}(A) = 0$, then $A = \boxplus_{i \in I} A_i$.

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1. *the X_i partition X*
2. $A = \Box_{i \in I} B_i$

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What is the 'best' sum decomposition?

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Let $A = \Box_{i \in I} A_i$ be a primitive axial algebra of Monster type. The finest sum decomposition of A is given by the connected components Δ_i of Δ.

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Let $A = \boxplus_{i \in I} A_i$ be a primitive axial algebra of Monster type. The finest sum decomposition of A is given by the connected components Δ_i of Δ. i.e. The finest axial sum decomposition is $A = \boxplus_{i \in I} B_i$ where $B_i := \langle\langle \Delta_i \rangle\rangle$.

Results

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Theorem

Let $A = \square_{i \in I} A_i$ *be an axial algebra. Then,* $\mathrm{Miy}(X)$ *is a central product of the subgroups* $\{\mathrm{Miy}(\Delta_i)\}_{i \in I}$*.*

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Aside: We say an algebra is m-closed if it is spanned by products of length at most m in the axes. So every algebra which is at most 3-closed is slender. We know of only a handful of algebras which are 4-closed or more, with the largest being 5-closed.

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Theorem

Suppose that A has a Seress fusion law. Then the conjecture holds if at most one $A_i := \langle\langle \Delta_i \rangle\rangle$ is not slender.

Classifications of axial algebras

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	1	0
1	1	
0		0

	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Figure: Fusion laws $\mathcal{A}$, $\mathcal{J}(\eta)$, and $\mathcal{M}(\alpha, \beta)$

The most simple fusion law is the following:

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Table: The fusion law $\mathcal{A}$

Recall that the 2-dimensional algebra $2\mathrm{B} = \langle\langle a, b \rangle\rangle$ satisfied this fusion law.

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Proposition

Let A be a primitive axial algebra. Then A is associative if and only if it satisfies the fusion law $\mathcal{A}$.

Jordan type

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Theorem (Hall, Rehren, Shpectorov, 2015)

Every primitive axial algebra of Jordan type $\eta \neq \frac{1}{2}$ is a Matsuo algebra $M_\eta(G)$.

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- However, every Jordan algebra which is 3-generated is also a Matsuo algebra (De Medts, Rehren with a correction from Yabe).
- For 4-generated algebras, this is no longer the case.
- New ideas of Gorshkov and Staroletov can be used to try to attack this case (plan of a workshop to be held).

Monster type $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$

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Theorem (Norton-Sakuma Theorem, Hall, Rehren, Shpectorov, 2015)

Let A be a 2-generated axial algebra of Monster type $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ over a field of characteristic 0 which admits a Frobenius form. Then A is one of the nine following algebras:

$$1\mathrm{A}, 2\mathrm{A}, 2\mathrm{B}, 3\mathrm{A}, 3\mathrm{C}, 4\mathrm{A}, 4\mathrm{B}, 5\mathrm{A}, \textit{or } 6\mathrm{A}$$

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Conjecture (Straight Flush Conjecture, Ivanov)

Suppose A is an indecomposable Majorana algebra where every number, 2, 3, 4, 5 *and* 6 *occurs in some* 2*-generated subalgebra. Then A embeds into the Griess algebra.*

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These new families don't all exist for every value of (α, β). Some exist for only a subvariety of $\mathbb{F}^2$.

Amazingly, if you plot the varieties where the algebras exist, they intersect in a unique point $(\frac{1}{4}, \frac{1}{32})$!

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3. *one of the algebras listed below:*
   1. $3\mathrm{A}(\alpha, \beta)$, $4\mathrm{A}(\frac{1}{4}, \beta)$, $4\mathrm{B}(\alpha, \frac{\alpha^2}{2})$, $4\mathrm{J}(\alpha, \frac{\alpha}{2})$, $4\mathrm{Y}(\frac{1}{2}, \beta)$, $4\mathrm{Y}(\alpha, \frac{1-\alpha^2}{2})$, $5\mathrm{A}(\alpha, \frac{5\alpha-1}{8})$, $6\mathrm{A}(\alpha, \frac{-\alpha^2}{4(2\alpha-1)})$, $6\mathrm{J}(\alpha, \frac{\alpha}{2})$, *or* $6\mathrm{Y}(\frac{1}{2}, 2)$*;*
   2. $\mathrm{IY}_3(\alpha, \frac{1}{2}, \mu)$, *or* $\mathrm{IY}_5(\alpha, \frac{1}{2})$.

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Franchi, Mainardis, M^{c}Inroy, (2021?) have classified all ideals and the surprise is that for every $n \in N$, there is a quotient with n axes.

Open question

Problem

What about 2-generated algebras of Monster type $\mathcal{M}(\alpha, \beta)$ which are not symmetric?

Larger algebras

For every pair of axes a and b in an axial algebra A, we have a 2-generated subalgebra $\langle\langle a, b \rangle\rangle$.

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(Aside: McInroy and Shpectorov have designed an algorithm and implemented it in Magma to calculate an axial algebra with a given Miyamoto group and shape.)

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Theorem (M^c Inroy and Shpectorov, 2021)

A 2-generated axet for a C_2-graded fusion law is one of two types:

1. *regular – either one orbit of axes, or two of the same length; or*
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We do not know of any examples of algebras with a skew axet. Such an example would necessarily be non-symmetric. Do such examples exist?

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Misaotra anao nihaino!

Merci de votre attention!